COUNTING INTEGER PARTITIONS: EXERCISE SHEET

Easier exercises are marked (*) and harder ones are marked (**). They can be done in any order. Do not hesitate to ask you teaching assistant for advice :)

Throughout this sheet, we denote by $p(n \mid \text{condition } C)$ the number of partitions of n satisfying condition C.

Exercise 1 (*).

(1) List all partitions of 6.
(2) How many of these have only even parts? How many of them are such that each part appears an even number of times? What do you notice?
(3) Explain why for n odd,

$$p(n \mid \text{even parts}) = p(n \mid \text{each part appears an even number of times}) = 0.$$

(4) Give a bijection to prove that for all $n \in \mathbb{N}$,

$$p(n \mid \text{even parts}) = p(n \mid \text{each part appears an even number of times}).$$

Exercise 2 (*). You want to buy a €1,99 candy bag at the supermarket and pay it in cash with exactly the right amount. How many possibilities do you have?

Reminder (especially for non-european students): In euros, we have the following coins: 1 cent, 2 cents, 5 cents, 10 cents, 20 cents, 50 cents, 1 euro, 2 euros.

Hint: It is recommended to use a computer algebra program if you have one, or kindly ask your TA to do it for you otherwise (you would have to tell them which generating function to compute and which coefficient to extract).

Exercise 3 (*). Use Ferrer diagrams to prove the following q-series identity:

$$\prod_{k\geq 1}(1+zq^k) = \sum_{n\geq 0} \frac{z^n q^{n(n+1)/2}}{\prod_{k=1}^{n}(1-q^k)}.$$

Exercise 4 (**). A partition is said to be *self-conjugate* if it is its own conjugate (for example, $5+3+2+1+1$ is self-conjugate).

(1) Draw the Ferrer diagrams of the three self-conjugate partitions of 12.
(2) Use a bijection to prove that for all $n \in \mathbb{N}$,

$$p(n \mid \text{self-conjugate}) = p(n \mid \text{distinct odd parts}).$$

Exercise 5 (**). Let k be a fixed positive integer. Prove the following identity:

for all n, the number of partitions of n such that no part is divisible by k equals the number of partitions of n where each part appears at most $k-1$ times

(1) using generating functions,
(2) bijectively.

Exercise 6 (**). Let n and m be non-negative integers. The q-binomial coefficient $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$ is defined to be the generating function for partitions whose Ferrer diagram fits inside a $n \times m$ rectangle (where the power of q counts as usual the number which is partitioned) We take the convention that it equals 0 when $m < 0$ or $n < 0$, and that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$.

(1) Compute $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$ and draw the corresponding Ferrer diagrams.
(2) Show that for all m and n,
$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n \\ n \end{bmatrix}_q.$$
(3) Using Ferrer diagrams, show that for all $m, n \geq 1$,
$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ m-1 \end{bmatrix}_q + q^m \begin{bmatrix} m+n-1 \\ m \end{bmatrix}_q.$$
Does this ring a bell?
(4) Using the previous question, show that for all m and n,
$$\begin{bmatrix} m+n \\ m \end{bmatrix}_1 = \binom{m+n}{m},$$
the usual binomial coefficient.
(5) Prove the q-binomial theorem:
$$\prod_{k=1}^{n} (1 + zq^k) = \sum_{m=0}^{n} z^m q^{m(m+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q.$$
This formula can be seen as a finite version of Exercise 3.
(6) What happens when you set $q = 1$ in the previous formula?

Exercise from the course (**). Let $d(n,k)$ denote the number of partitions into distinct parts. Using generating functions, show that for all $k \geq 1$ and $n \geq k$,
$$d(n,k) = d(n-k,k) + d(n-k,k-1).$$
We take the convention that $d(0,0) = 1$ (the empty partition), and that $d(n,k) = 0$ for all $n < 0$ and whenever $k > n$.

Use this recurrence to compute $d(10,3)$.